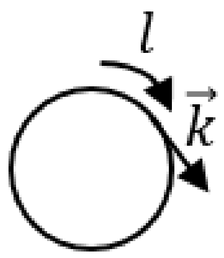
3. Landau Levels and Shubnikov-de Haas oscillations

First thing to do is to solve the Hamiltonian and to find the Eigen energies.

i. Landau Levels in a semi-classical approach

In quantum mechanics, the charge carrier ↔ complex wave function $\psi(\vec{r})$ with a phase $\varphi(\vec{r})$.
charge in 2D, no disorder but with a magnetic field

<u>semi-classical approach</u>: *classical trajectories with a quantum phase that accounts for the circulation along the cyclotron orbital*



In the presence of a magnetic field, the total phase acquired by a quasi-particle will also be given by the circulation of the potential vector $\vec{A} = \frac{Br}{2}\overrightarrow{u_\theta} = \frac{BR_c}{2}\overrightarrow{u_\theta}$ and we end up with

$$\varphi = \oint \left(\vec{k} + q\vec{A}(r = R_c)/\hbar\right) \cdot \overrightarrow{dr}$$

According to Onsager and to the Bohr-Sommerfeld quantification rules, the phase acquired on a closed cyclotron orbital should satisfy: $\varphi = 2(n+\gamma)\pi$ with $n \in \mathbb{N}$ and $\gamma \in \mathbb{R}$.

As the scalar products $\vec{k} \cdot \overrightarrow{dr}$ and $\vec{A} \cdot \overrightarrow{dr}$ are constant all along the orbital, we end up for an electron ($q = -e$) with:

$$\varphi = (k - eBR_c/2\hbar) \times 2\pi R_c = 2\pi(n+\gamma)$$

With $R_c = v/\omega_c = \hbar k/m\omega_c = \hbar k/eB$

Finally, we have

$$k/2 \times \hbar k/m\omega_c = n+\gamma$$

$$\hbar k^2/2m\omega_c = n+\gamma$$

$$\hbar^2 k^2/2m = \hbar\omega_c(n+\gamma)$$

It is not possible to determine the value of γ in a semi-classical approach but the quantum mechanics tell us that $\gamma = 1/2$.

In the semi-classical approach, we consider that the magnetic field only change the trajectories and not the general expression of the Hamiltonian so that we have to consider the case of a Fermi sea with massive free electrons. As a result, the energy of a massive charge coupled to a magnetic field will be still given (only in our semi-classical approach) by $E_n = \hbar^2 k^2/2m$.

We finally find the Landau solution of the Hamiltonian for a free electron embedded in a magnetic field:

$$E_n = \hbar\omega_c\left(\frac{1}{2} + n\right)$$

The consequence of switching on the magnetic field will be the quantification of the energy.

due to interferences along orbitals, we open gaps in the density of states.

In a similar way than for the classical case:

pure material

$\Leftrightarrow$

material with weak enough disorder (or the magnetic field high enough) to allow the formation of cyclotron orbitals

$\Leftrightarrow$

$R_c < l_e$

ii. Quantum mechanics treatment

In a general way, we have: $\mathcal{H}|\psi\rangle = i\hbar\frac{\partial}{\partial t}|\psi\rangle$

which can be simplified if $\frac{\partial}{\partial t}\mathcal{H} = 0$

We have then

$$\mathcal{H}|\psi\rangle = E|\psi\rangle$$

For a Fermi sea of massive Fermions without magnetic field,

$$\mathcal{H} = \frac{(\hbar k)^2}{2m}$$

If the magnetic field $\vec{B} = \vec{\nabla} \wedge \vec{A}$ is turned on → Peierl's substitution: $m\vec{v} = \hbar\vec{k} \leftrightarrow \vec{p} = \hbar\vec{k} + q\vec{A}$ with $\vec{p} = -i\,\hbar\vec{\nabla}$.

The Hamiltonian that has to be solved in the case of electrons is:

$$\frac{\left(\hbar\vec{k}\right)^2}{2m}|\psi\rangle = \frac{\left(\vec{p} + e\vec{A}\right)^2}{2m}|\psi\rangle = E|\psi\rangle$$

This is a classical problem of the quantum mechanics. This have been solved by **Lev Landau** in 1930 (he was 21 years old!). As suggested by the semi-classical model, gaps open in the energy spectrum and only some discrete values of the energy will be allowed:

$$E_n = \hbar\omega_c\left(\frac{1}{2}+n\right)$$

The major achievement of the quantum mechanics is to show the existence of a **zero point energy** $E_0 = \frac{\hbar\omega_c}{2}$ that is a direct consequence of the Heisenberg inequality $\Delta x_0 \Delta k_0 \sim \hbar$. It is a fundamental property and we will see that this zero energy points is not present for massless Dirac Fermions like graphene quasi-particles!

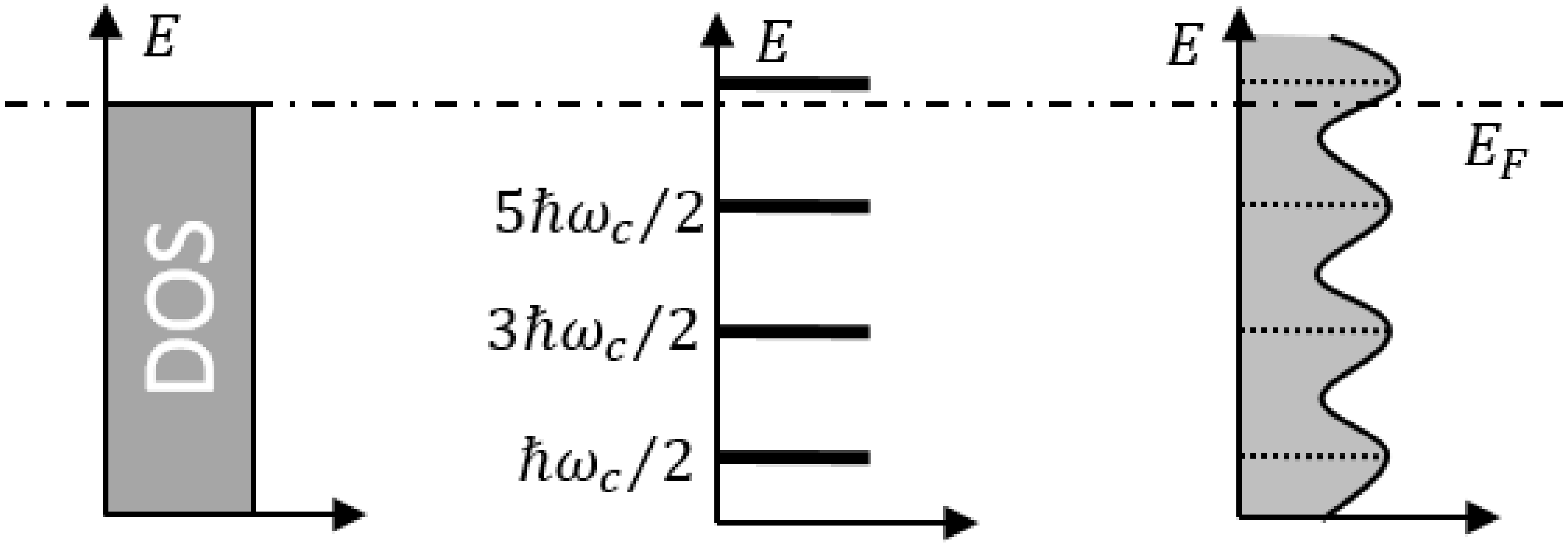


Fig. 4 Density of states at 2D for a zero magnetic field, for a non-disordered 2D metal and for a disordered 2D-metal

When the magnetic field is high enough to open some gaps into the DOS, the transport properties are determined by the position of the Fermi energy E_F.

For an infinite sample, we have the relation

$$\sigma = e^2 \times DoS(E_F) \times D = e^2 \times DoS(E_F) \times v_F l_e/d)$$

- If E_F lies between to Landau Levels (LL), the material will be insulating as predicted by the classical model

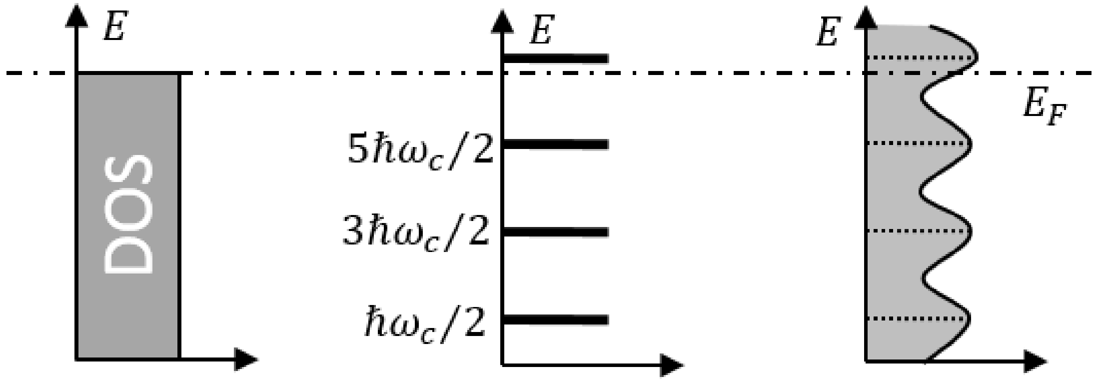
- If E_F is pined on a LL, the density of states is ≠ 0 at E_F so that transport should occurs. Nevertheless, even an infinitesimal disorder will localize those states such that $l_e = 0$. The sample will also be in an insulating state!

In the case of an infinite 2D metal (with no edges) with an infinitesimal disorder, the **switching of a magnetic field induces a transition to an insulating state** as predicted by the classical approach!

This simple picture does not take into account the effect of the edges (in the QHE regime, very important!)

Does not take into account the effect of the disorder that induces ***broadening of the LL***.

In the regime where the broadening induced by the disorder is larger than the energy spacing between two LLs $\Delta E = \hbar\omega_c$, magnetic field is not strong enough to open gaps into the energy spectrum and the DoS does not vanish for any energies **: as in the classical case, the disorder restores the conduction into the bulk!**



Nevertheless, the magnetic field induces a fluctuation of the DOS.

DOS exhibits maximum at energies corresponding to a LL as soon as the magnetic field is strong enough to close the cyclotron orbitals $(R_c \lesssim l_e)$.

Such a fluctuating DOS has some strong implication for the transport properties of a the metals: for a given sample, the position of the Fermi energy E_F is fixed by the doping level in the sample. Sweeping the magnetic field shifts the position of the LL with respect to E_F since $\hbar\omega_c \propto B$ and **the $\boldsymbol{DOS(E_F)}$ fluctuates as well as the conductance of the sample.**

A minimum of the conductance (at a magnetic field B_{min}^n) corresponds to the crossing of a LL.

Such oscillations, called **Shubnikov-de Haas (SdH) oscillations**, are very useful to determine:

- the **charge carrier density** of a 2D metal (thanks to the value of the different B_{min}^n),
- the **effective mass** (the thermal dependence of $\Delta G_{min}(T)$) and
- the **mean free path** (the dependence of $G_{min}(B_{min}^n)$ with respect to the magnetic field or the onset of the oscillations that corresponds to $R_c \lesssim l_e \Leftrightarrow \frac{1}{\tau} \lesssim \omega_c \Leftrightarrow \frac{1}{\mu} \lesssim B$).

iii. Temperature dependence of SdHO

Not only the mobility is required to be good for the observation of SdH oscillations ($\mu B \gtrsim 1$).

The temperature needs also to be low enough: the inequality: $\hbar\omega_c \gtrsim k_B T$ has to be satified.

For usual materials such as GaAs and standard Magnetic fields (few T), this means that $T \lesssim 10K$.

For graphene, the effective mass is much smaller and $\rightarrow 0$ so that the cyclotron energy is much larger and the temperature requirement for the observation of SdH oscillations becomes $T \lesssim 1000K$: **SdHO can be observed at room temperature in graphene**!

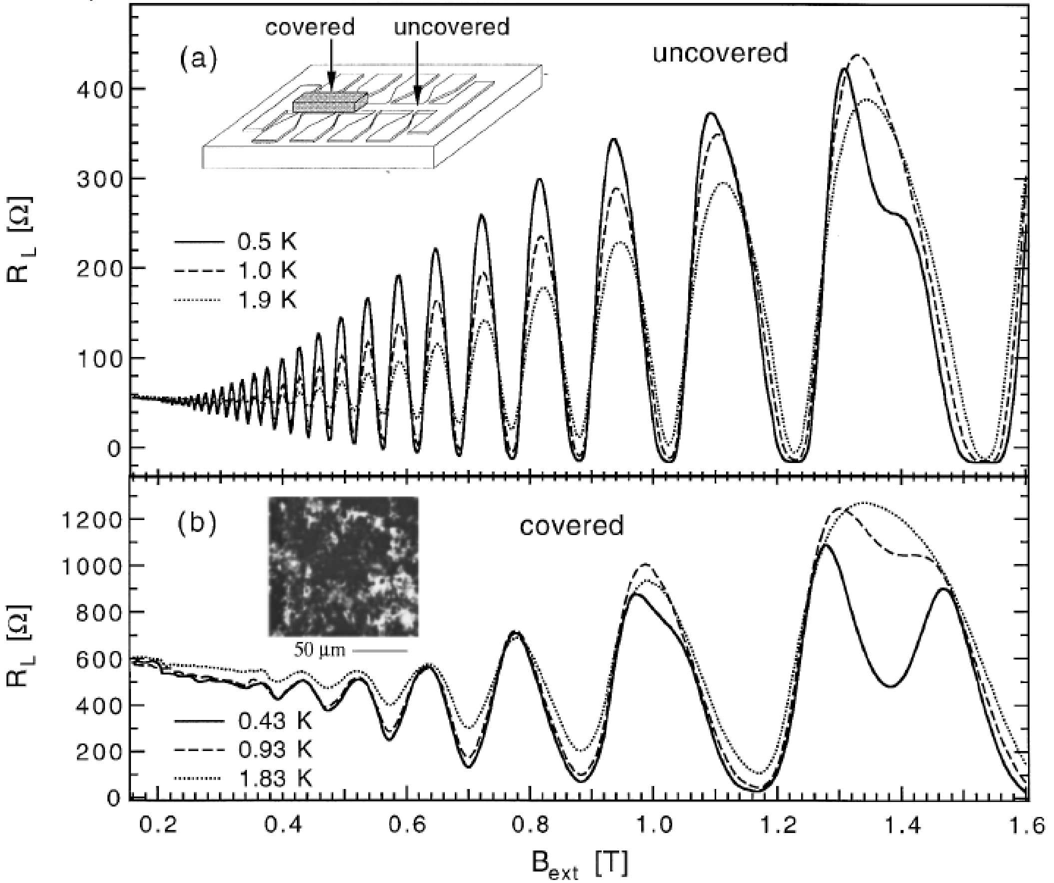


Fig. 5 Example of SdH oscillations measured in GaAs based heterostructures at low temperature (from Malcoff et al.)

iv. SdHO for massless fermions

In a very general case, it can be shown that the parameter that determines the temperature evolution of the SdH oscillations will be the **cyclotron mass** that is expressed, in the case of masseless fermions as:

$$m_{cycl} = \frac{E_F}{{v_F}^2}$$

For massive particles, the energy is dominated by the term $E = mc^2$ which leads to an m_{cycl} that is independent on the charge carrier density.

For relativistic massless particles in graphene, we have:

$$m_{cycl} = \frac{v_F \hbar k_F}{{v_F}^2} = \frac{\hbar}{v_F} k_F = \frac{\hbar\sqrt{\pi}}{v_F}\sqrt{n}$$

The cyclotron mass $\boldsymbol{m_{cycl}}$ depends on the charge carrier density in the case of massless fermions and is independent on then charge carrier density for massive charge carriers.

The measurement of the temperature dependence of the SdH oscillations and **the study of the density dependence of the cyclotron mass gives an unambiguous proof of the nature (classical or relativistic) of the charge carriers that are involved in the transport properties of a sample.**

$$m_{cycl} = \frac{\hbar\sqrt{\pi}}{v_F}\sqrt{n}$$

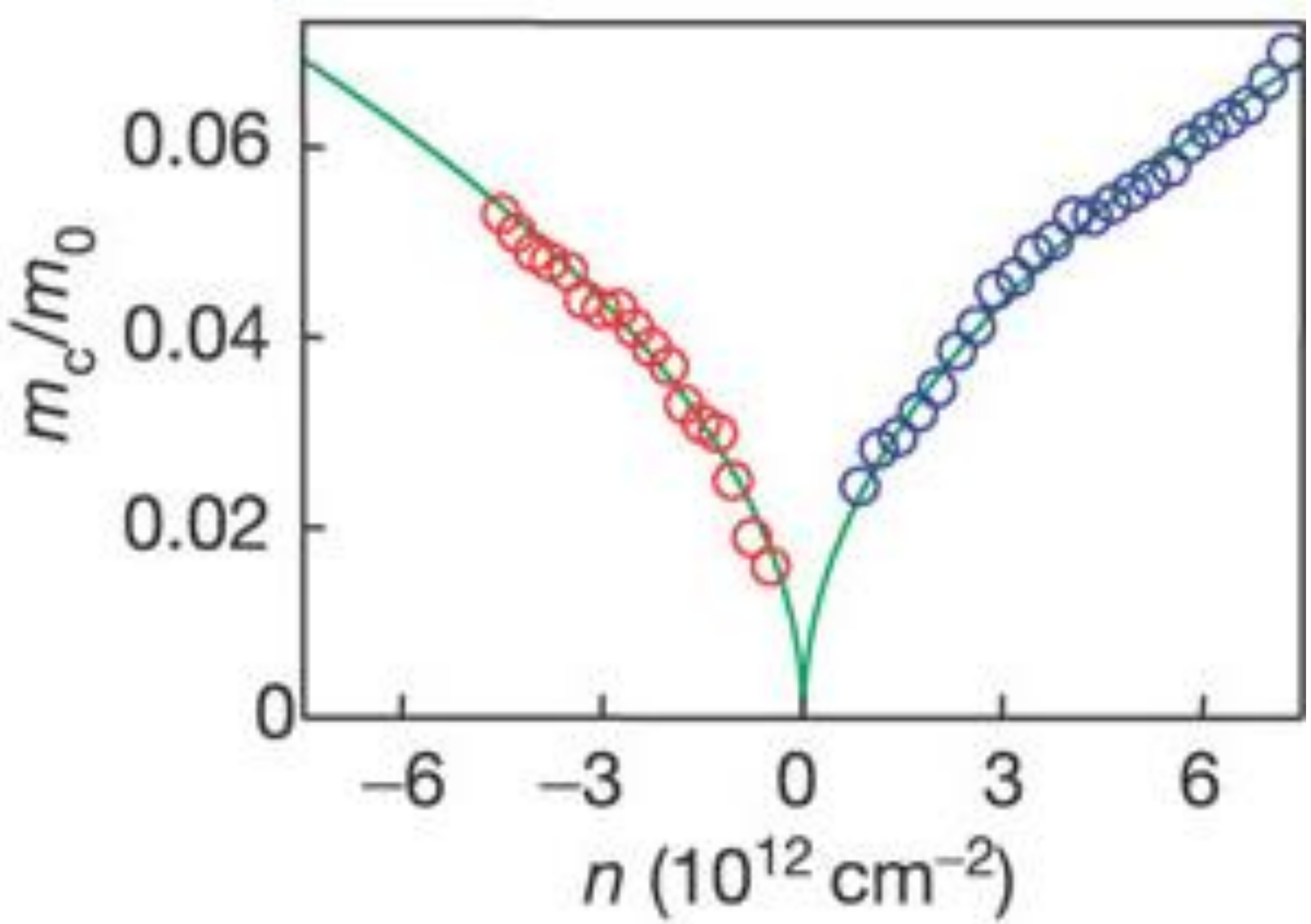
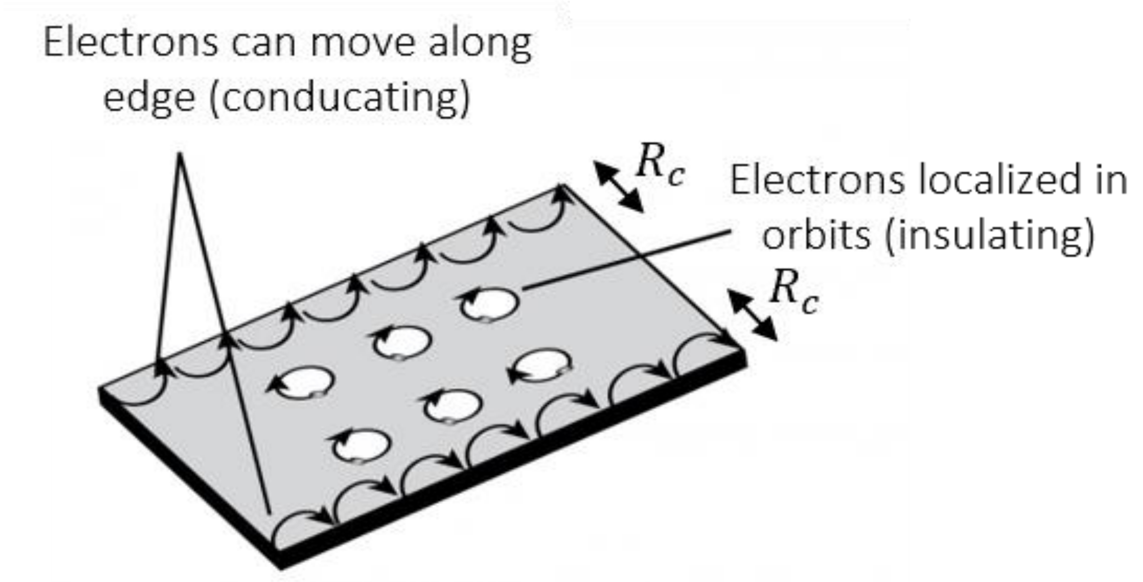


Fig. 7 Example of the measurement of the cyclotron mass in a graphene sheet. The mass is found to depend on the charge carrier density n like $n^{1/2}$ which an unambiguous proof that charge carriers in graphene are massless (from Novoselov et al.)

5. The quantum Hall effect in non-relativistic fermions (electrons)

We have seen that the classical model as well as the quantum mechanics expect both an insulating transition when the magnetic field is turned on for an infinite 2D metal with no disorder. As we will see, the introduction of edges drastically changes the conclusion of both models.

i. Classical model of a finite (with edges) 2D-metal with no disorder



In the classical model, the "bulk" remains insulating but we have the formation of conducting stripes which width is about R_c. Those channels should not be back-scattered and should perfectly conduct the current. Very importantly, the transport is chiral.