# Derivation of the renormalization group equations of `cond-mat/0501022`

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These notes constitute additional material for the paper entitled "Quantum electrodynamics in $2+1$ dimensions, confinement, and the stability of $U(1)$ spin liquids", by the same authors, which appeared in the archive under number `cond-mat/0501022`. In these notes we make a thorough derivation of the renormalization group equations exhibited in our work.

## I. KOSTERLITZ'S RENORMALIZATION GROUP EQUATIONS FOR THE $d$-DIMENSIONAL COULOMB GAS

We start by deriving Eqs. (6) and (7) of our paper [1], corresponding to the renormalization group (RG) equations for the three-dimensional Coulomb gas of magnetic monopoles. We will consider here the general case of a $d$-dimensional Coulomb gas, whose RG equations were first obtained by Kosterlitz [2]. In our paper [1] the only dimensionful parameter available is $e_0^2$, which has dimension of inverse length in three dimensions. Instead of working with this quantity, we shall introduce another bare parameter, $K_0$, having dimension of length to the power $d-2$. For $d = 3$, which is the case of interest in our paper, we will set $K_0 \equiv 1/e_0^2$.

The RG equations for $d$-dimensional Coulomb gas were originally derived by Kosterlitz [2] using the so called poorman scaling approach. Here we will use a method due to Young [3], which is physically appealing, since it just amounts to perform a scale-dependent Debye-Hückel theory, and leads to exactly the same results. Although Young applied the method to derive the RG equations associated to the Kosterlitz-Thouless (KT) phase transition, it generalizes easily to the $d$-dimensional case. We have used this method previously [4] to derive the RG equations for anomalous Coulomb gases in $d$-dimensions. Here we concentrate on the ordinary $d$-dimensional Coulomb gas and try to keep the discussion self-contained and, hopefully, pedagogic.

The bare Coulomb interaction is given by

$$U_0(r) = -4\pi^2 K_0 V(r), \tag{1}$$

where

$$V(r) = \frac{a^{2-d}}{4\pi^{d/2}} \Gamma\left(\frac{d}{2} - 1\right) \left[\left(\frac{r}{a}\right)^{2-d} - 1\right]. \tag{2}$$

In the above equation $a$ is a short distance cutoff, which for $d = 3$ will be set to $a = 1/e_0^2$. From Eq. (1) we obtain the bare electric field:

$$E_0(r) = -4\pi^2 c(d) \frac{K_0}{r^{d-1}}, \tag{3}$$

where

$$c(d) = \frac{d-2}{4\pi^{d/2}} \Gamma\left(\frac{d}{2} - 1\right). \tag{4}$$

Next, in the spirit of the Debye-Hückel theory, we introduce an effective medium via a scale-dependent dielectric constant $\varepsilon(r)$. This gives the renormalized electric field,

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$$E(r) = -4\pi^2 c(d) \frac{K_0}{\varepsilon(r) r^{d-1}}. \tag{5}$$

The dielectric constant is given by

$$\varepsilon(r) = 1 + S_d \chi(r), \tag{6}$$

where $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface of the $d$-dimensional unit sphere, and the susceptibility is given by

$$\chi(r) = S_d \int_a^r ds s^{d-1} \alpha(s) n(s). \tag{7}$$

In Eq. (7) $\alpha(r)$ is the polarizability, which for small separation of a dipole pair is given approximately by

$$\alpha(r) \approx \frac{4\pi^2 K_0 r^2}{d}. \tag{8}$$

The average number of dipole pairs is approximately given by

$$n(r) \approx z_0^2 e^{-U(r)}, \tag{9}$$

where $z_0$ is the bare fugacity and $U(r)$ is the renormalized potential obtained by integrating the renormalized electric field (5),

$$U(r) = U(a) + 4\pi^2 c(d) K_0 \int_a^r \frac{ds}{s^{d-1} \varepsilon(s)}. \tag{10}$$

The renormalized counterpart of $K_0$ is given by

$$\frac{1}{K(l)} = \frac{\varepsilon(ae^l)}{K_0} e^{(d-2)l}, \tag{11}$$

where $l = \ln(r/a)$. Differentiation of Eq. (10) with respect to $l$ gives

$$\frac{dU}{dl} = \frac{4\pi^2 c(d)}{a^{d-2}} K(l), \tag{12}$$

where we have used Eq. (11). Next we differentiate Eq. (11) with respect to $l$ to obtain

$$\frac{dK^{-1}}{dl} = \frac{8\pi^2 S_d^2 z_0^2 a^{d+2}}{d} e^{2dl - U(ae^l)} + (d-2)K^{-1}. \tag{13}$$

Now we define

$$z^2(l) = \frac{2S_d^2 z_0^2}{d} e^{2dl - U(ae^l)}, \tag{14}$$

such that Eq. (13) becomes

$$\frac{dK^{-1}}{dl} = 4\pi^2 a^{d+2} z^2 + (d-2)K^{-1}. \tag{15}$$

From Eq. (14) we derive the RG equation for the effective fugacity:

$$\frac{dz}{dl} = \left[ d - \frac{2\pi^2 c(d)K}{a^{d-2}} \right] z. \tag{16}$$

It is convenient to introduce the dimensionless couplings $\kappa \equiv a^{2-d}K$ and $y \equiv a^d z$ to rewrite Eqs. (15) and (16) as

$$\frac{d\kappa^{-1}}{dl} = 4\pi^2 y^2 + (d-2)\kappa^{-1}, \tag{17}$$

$$\frac{dy}{dl} = \left[ d - 2\pi^2 c(d)\kappa \right] y. \tag{18}$$

For $d = 2$ the above RG equations govern the scaling behavior of the KT transition, while for $d > 2$ there is no fixed point, implying that the $d$-dimensional Coulomb gas is always in the metallic phase [2].

In the context of compact Maxwell theory in $d = 3$, we set $K_0 = a = 1/e_0^2$ and define the dimensionless gauge coupling $f \equiv K_0/K = 1/\kappa = e^2/e_0^2$. In this way we obtain from Eqs. (17) and (18) the RG equations for compact Maxwell theory in three space-time dimensions:

$$\frac{df}{dl} = 4\pi^2 y^2 + f, \tag{19}$$

$$\frac{dy}{dl} = \left( 3 - \frac{\pi^3}{f} \right) y. \tag{20}$$

## II. COUPLING TO FERMIONIC MATTER FIELDS: THE RG EQUATIONS FOR COMPACT QED3

For $d = 3$ and in the absence of matter, Eq. (11) gives the renormalization of the bare charge due to the magnetic monopoles, as can be more easily seen after bringing it to the form

$$e^2(l) = \varepsilon(ae^l)e^l e_0^2. \tag{21}$$

In the non-compact case, the bare charge is renormalized by the wave function renormalization of the gauge field, $Z_A(l)$, which is calculated from the vacuum polarization. Thus, for non-compact QED3 we have simply $e^2(l) = Z_A(l)e^l e_0^2$. In order to account for the effects of matter fields we have to consider in the compact case the renormalization due to the vacuum polarization. In such a case Eq. (21) is modified to

$$e^2(l) = Z_A(l)\varepsilon(ae^l)e^l e_0^2. \tag{22}$$

Differentiation of Eq. (22) with respect to $l$ yields

$$\frac{de^2}{dl} = \frac{128\pi^4 z_0^2}{3(e_0^2)^5} Z_A(l)e^{6l - U(e^l/e_0^2)} + (1 - \gamma_A)e^2(l), \tag{23}$$

where

$$\gamma_A \equiv -\frac{d\ln Z_A}{dl}. \tag{24}$$

From Eq. (23) we see that the quantity $z^2$ also gets modified by the gauge field wave function renormalization. Note that in all equations of Section I the "dielectric" constant is multiplied by $Z_A(l)$. After defining the dimensionless fugacity

$$y^2 = \frac{32\pi^2 z_0^2}{3(e_0^2)^6} Z_A(l)e^{6l - U(e^l/e_0^2)}, \tag{25}$$

and setting $f = e^2/e_0^2$ we obtain the RG equation for the dimensionless gauge coupling,

$$\frac{df}{dl} = 4\pi^2 y^2 + (1 - \gamma_A)f. \tag{26}$$

Finally, differentiation of Eq. (25) with respect to $l$ leads to the RG equation

$$\frac{dy}{dl} = \left(3 - \frac{\pi^3}{f} - \frac{\gamma_A}{2}\right)y. \tag{27}$$

It is straightforward to calculate $\gamma_A$ at one-loop order,

$$\gamma_A = \frac{Nf}{8}. \tag{28}$$

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