Local coordinates on a real manifold $M$ of dimension $m$:
$U \subset M$ is an open set small enough to be trivialized:

$$M \supset U \ni x \mapsto \tilde{x} = (x^1, \ldots, x^m) \in \mathbb{R}^m$$

Élie Cartan’s external calculus on $M$ does not need a metric:
- external 0-form: $f : x \mapsto f(x) = \tilde{f}(\tilde{x}) \in \mathbb{R}$,
- external 1-form: $\omega = df = (\partial f/\partial x^i) dx^i = \sum \omega_j dx^j$,
- external 2-form: $\sigma = \sum_{i \leq j} \sigma_{ij} dx^i \wedge dx^j = (2!)^{-1} \sigma_{ij} dx^i dx^j$, 
- external $r$-form: $\rho = \sum_{i_1 < \ldots < i_r} \rho_{i_1 \ldots i_r} dx^{i_1} \wedge \ldots \wedge dx^{i_r} = (r!)^{-1} \rho_{i_1 \ldots i_r} dx^{i_1} \ldots dx^{i_r}$
  - alternating tensor $\rho_{i_1 \ldots i_r}$.

$y = y(x) : df = (\partial f/\partial x^i) dx^i = (\partial f/\partial y^j)(\partial y^j/\partial x^i) dx^i = (\partial f/\partial y^j) dy^j.$
An action integral over $M$ which is an integral of an external form depends only on the topology of $M$, not on a metric; it is called a topological term.

For instance the action integral of a Maxwell field,

$$\int d^4x F_{\mu\nu} F^{\mu\nu} = \int d^4x F_{\sigma\tau} g^{\sigma\mu} g^{\tau\nu} F_{\mu\nu}$$

depends on the pseudo-Riemannian metric $g^{\mu\nu}$, while

$$\int d^4x \delta_{0123} F_{\mu\nu} F_{\rho\sigma} = \int d^4x E \cdot B$$

is a topological term.

Theoretical prediction of quantum oscillations in $\sigma_{xy}$:

$$\sigma_{xy} \approx \frac{\text{neq}}{B} + \frac{1}{\omega_c^2} \sigma_{xx}.$$  


Discovery of QHE:

Kubo's formula as a Chern number: (TKNN-number)


A case of combinatorial topology:

Relevance in gauge-field theory and string theory:

In this paper Witten coined the names Chern-Simons field theory and Chern-Simons action.

(2 + 1)-dimensional Chern-Simons field theory:

\[ S_{\text{int}} = \int d^3x \frac{\sigma_H}{2i} \sum_{\mu=0}^2 \int d^3x \frac{\delta \mu_{012}}{012} \partial_\mu A_\nu. \]

\[ \sigma_H \frac{\delta S_{\text{int}}}{\delta A_\nu} = \sigma_H \frac{\delta \mu_{012}}{012} \partial_\mu A_\nu, \]

\[ \mu = 0, 1, 2; \]

\[ j^\mu = \frac{\delta S_{\text{int}}}{\delta A_\nu} = \sigma_H \frac{\delta \mu_{012}}{012} \partial_\mu A_\nu, \]

\[ \partial_\mu = \frac{\partial}{\partial x^\mu}, \]

\[ \mu = 0, 1, 2; \]

\[ j^0 = \sigma_H F^0, \quad F^0 = \frac{\delta \mu_{012}}{012} \partial_\mu A_\nu, \]

\[ j^1 = \sigma_H E^2, \quad j^2 = -\sigma_H E^1. \]

\[ \Rightarrow \frac{\partial B_z}{\partial t} = -\nabla \times E: \quad \text{Faraday's law of induction.} \]
Kubo:

\[ \sigma_H = \frac{e^2}{\hbar} \left( \frac{1}{2\pi^2} \int \frac{d^2k \, \delta(k)^2}{\hbar^2} \right) \int \sum_{n} (-2i) \left( \partial_i u_{mn} \right) \frac{d\kappa_1 \wedge d\kappa_2}{\partial_{\kappa_j}} \right) \]

integer (first Chern number)

\[ \sigma_H = \frac{e^2}{\hbar} \left( \frac{1}{2\pi^2} \int \frac{d^2k \, \delta(k)^2}{\hbar^2} \right) \int \sum_{n} (-2i) \left( \partial_i u_{mn} \right) \frac{d\kappa_1 \wedge d\kappa_2}{\partial_{\kappa_j}} \right) \]

A = \sum_i A_i \, d\kappa_i : Berry-Simon connection form

\[ F = DA = dA + i[A, A] = \left( \text{tr}([A, A]) = 0 \right) \]

\[ \int \sum_{\delta} \partial_i A_i \, d\kappa_i \wedge d\kappa_j : \text{Berry-Simon curvature form} \]

q = (q_A) = (\alpha, x^1, x^2, \kappa_1, \kappa_2), \quad A = 0, 1, 2, 3, 4, \quad \partial_A = \frac{\partial}{\partial q_A} \cdot \]

\[ (A_k) = (A_0, A_1, A_2, 0, 0), \quad (A_A) = (0, 0, 0, A_3, A_4), \]

\[ S_{\text{int}} = \frac{\sigma^2}{h} \left( \frac{1}{2\pi^2} \int d^2q \, \delta^{ABCDEF} A_{\text{tr}} (A_{\partial A}) \right). \]

integer rth Chern number \( C_r \) of rth Chern character:

\[ C_r = \frac{1}{r!} \int \left( \frac{F}{2\pi} \wedge \cdots \wedge \frac{F}{2\pi} \right) = \frac{1}{r!} \int \left( \frac{2}{2\pi} \int d^2k \, F^{12} \right)^r = \]

\[ \frac{1}{r!} \int \left( \frac{2}{2\pi} \int d^2k \, \delta^{12} \frac{2}{3} \right)^r \int (F^1 \wedge F^2 \wedge \cdots \wedge F^{r+1}) \].

\[ S_{\text{int}}^{(2r+1)} = \frac{\sigma^2}{h} \left( \frac{C_r}{(2\pi)^r} \right) \int A_0 \delta A_1 \cdots \delta_{2r-1} A_{2r} \, cdt \wedge dx^1 \wedge \cdots \wedge dx^{2r} = \]

\[ = \frac{\sigma^2}{h} \left( \frac{C_r}{(r+1)!} \right) \int d^{2r+1} x \, \delta^{012 \cdots 2r} A_0 \delta A_1 \cdots \delta_{2r} A_{2r}. \]

Preliminaries
Historical Remark
Quantum Hall Effect
(2r + 1)-Dimensional Chern-Simons Field Theory
Bloch and Wannier functions
Kubo’s formula
Dimension Reduction
Z2-Symmetry
Summary
rth order response:

\[ S^{(2r+1)}_{\text{int}} = \frac{e^2}{h} \sum_{r} \frac{1}{r! (2\pi)^r} \int \alpha^{r+1} q j_{\alpha} A_{\alpha} \cdot A_{\alpha} \cdot \ldots \cdot A_{\alpha}, \]

Simple (2r + 1)D model, topologically equivalent to most conceivable 2rD band structures:

(2r + 1)D Dirac model:

\[ H = \int d^2 \phi \phi^l \left( -i \sum_j \Gamma^j \partial_j + \Gamma^0 m \right) \psi, \quad [\Gamma^\mu, \Gamma^\nu]_+ = 2\delta^{\mu\nu} I, \]

where \( j = 1, \ldots, 2r, \quad \mu, \nu = 0, 1, \ldots, 2r. \)

The \( \Gamma^\mu \) are \( (r \times r) \)-matrices of a faithful representation of the \( \text{so}(2r + 1) \) Lie algebra (Clifford algebra).

2D Dirac lattice model (tb):

\[ \hat{H} = \sum_\mathbf{k} \frac{1}{2} \left( \sum_j \Gamma^j \sin k^j + \Gamma^0 (m + c \sum_j \cos k^j) \right) \hat{\psi}_k = \]

\[ = \sum_\mathbf{k} \frac{1}{2} \left( \sum_\mu \Gamma^\mu d_\mu(\mathbf{k}) \right) \hat{\psi}_k, \quad c \text{: NN hopping integral.} \]

Example: 4D QHE


\[ r = 2, \quad A_0 = A_1 = A_2 = 0, \quad A_3 = -E_3 \text{ct :} \]

\[ f^4 = \frac{e^2}{h} \frac{C_2}{(2\pi)2} B_5 E_3 = \sigma_{\text{II}} E_3, \]

\[ \int dx_1 dx_2 f^4 = \frac{e^2}{h} \frac{C_2}{(2\pi)^2} \left( \int dx_1 dx_2 B_5 \right) E_3 = \frac{e}{2\pi} C_2 N E_3, \]

Example: 4D QHE


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\[ \begin{aligned}
E_3 & \rightarrow \\
B_3 & \rightarrow \\
\end{aligned} \]
gapless surface states on 3D surface (open $x^4$ boundary) for $\varepsilon_k = 0$ for some $k$.

1 Dirac cone at $k = 0$ for $|C_2| = 1$,
3 Dirac cones at $k = (\pi, 0, 0), (0, \pi, 0), (0, 0, \pi)$ for $|C_2| = 3$.

For more details see Qi et al. (2008).
Consider a 1D crystal:
\[ R = N \mathbf{a}, \quad G = 2\pi \mathbf{a}, \quad |\mathbb{T}_1^1| = |\mathbf{a}|, \quad |\mathbb{T}_k^1| = \frac{2\pi}{|\mathbf{a}|} \]
\[ \theta(k) = \alpha(k) + Nk, \]
\[ \Delta P = \frac{e}{|\mathbb{T}_1^1|} \Delta \left( \frac{a}{2\pi} (-i) \int_0^1 a k \sum_{n} (\mathbf{u}_n \mathbf{a} \cdot \partial_k (\mathbf{u}_n \mathbf{a}) \right). \]

Helmut Eschrig IFW Dresden

Theory of Topological Insulators

Adiabatic parameter \( t \):
\[ \hat{H} = \hat{H}_t, \quad 0 \leq t \leq 1, \quad P(0) = 0, P(1) = P, \]
\[ u_{nk} \rightarrow u_{ntk}, \quad U(k) \rightarrow U(k, t). \]

\[ \partial M \]
\[ M \]
\[ k \]
\[ t \]
\[ \mathbb{T}_1^1 \]

Let \( \hat{H}_t \) be periodic, \( \hat{H}_t = \hat{H}_0 \), then, \( \partial M = \emptyset \). Is \( P(1) = P = 0 \)?

Recall that \( \Delta P \) was via \( \mathcal{A} \) only determined modulo \( (e/|\mathbb{T}_1^1|)Na \).
\[ \frac{1}{2\pi} \int_M \mathcal{F} = N = \text{integer} \quad (= \text{first Chern number } C_1). \]

While \( \text{tr}\mathcal{A} \) depends on \( U(k, t) \), \( \text{tr}\mathcal{F} \) depends only on \( U(k) \) (for smooth \( U(t) \)).

\[ \text{tr}\mathcal{F} = d\text{tr}\mathcal{A} \quad \text{holds locally, its continuation to all } M \text{ is obstructed, if } C_1 \neq 0. \]

This is a case of the general Chern-Weil theorem:
While \( \text{tr}\mathcal{F} \) is uniquely defined in such cases, it is independent of the local gauge \( (U(t)) \) of \( \text{tr}\mathcal{A} \).

A_x = 0, A_y = Bx – Ecc. \( B = Bez, E = Ee_y \). 
\[ \hat{H} = \frac{\hbar^2}{2m} \left( -i \frac{\partial}{\partial x} \right)^2 + \left( -i \frac{\partial}{\partial y} + \frac{e}{\hbar} Bx - \frac{e}{\hbar} E \right)^2 \right) + V(x, y) \]
\[ \hat{T}(\mathbf{R}) = \exp \left( iR \cdot (-i\nabla - (e/\hbar)\mathbf{e}_y Bx) \right), \quad \mathbf{R} = n_1 \mathbf{a}_1 + n_2 \mathbf{a}_2, \]
\[ \hat{T}(\mathbf{R}) \hat{T}(\mathbf{R}') = \exp \left( i(e/\hbar)\mathbf{B} \cdot (\mathbf{R} - \mathbf{R}') \right) \hat{T}(\mathbf{R}) \hat{T}(\mathbf{R}'), \quad [\hat{H}, \hat{T}(\mathbf{R})] = 0, \]
choose \( e/\hbar \mathbf{B} \cdot \mathbf{R}_1 \times \mathbf{R}_2 = 2\pi \cdot \text{integer} \), then
\[ \hat{H}_{\text{occ}} u_{\text{occ}} = u_{\text{occ}} \hat{H}_{\text{occ}}, \quad \hat{R}_{\text{occ}} = \frac{\hbar^2}{2m}(\mathbf{\hat{r}} + \mathbf{m})^2 + V, \quad \mathbf{\hat{r}} = -i\nabla - (e/\hbar)\mathbf{e}_y (Bx - Ecc). \]
\[ \hat{J} = -\frac{e\hbar}{m} \sum_{i=1}^{2} \hat{R}_{\text{occ}} \left[ \frac{\partial}{\partial y}, \hat{R}_{\text{occ}} \right]. \]
\[ \sigma_H = \frac{i\hbar}{(2\pi)^2} \int d^2k \sum_{n, m} \frac{(-2i)^n u_{\text{occ}} | \partial^2 u_{\text{occ}} \rangle d\kappa_1 \wedge d\kappa_2}{(\epsilon_{m\kappa} - \epsilon_{n\kappa})^2} \]


Conductivity \( \sim \) current-current correlation function; for non-interacting Bloch electrons at zero temperature:

\[ \sigma_H = \frac{\hbar}{(2\pi)^2} \int d^2k \sum_{n, m} \frac{(-2i)^n u_{\text{occ}} | \partial^2 u_{\text{occ}} \rangle d\kappa_1 \wedge d\kappa_2}{(\epsilon_{m\kappa} - \epsilon_{n\kappa})^2} \]
Remark regarding the literature:

\[ G^{-1} G = \frac{\partial G^{-1}}{\partial t} + G^{-1} \partial G = 0; \]

\[ \text{tr}(G^{-1} \partial G \cdots) = \mp \text{tr}(G \partial G^{-1} \cdots) \]

D. J. Thouless, Phys. Rev. B 27, 6083 (1983);

Consider a QHE setup with a strip long in y-direction (further on with periodic boundary conditions with macroscopic length \( L_y \)) but constraint in x-direction by width \( L_x \), and let the magnetic field adiabatically change.

\[ A_0 = 0, \quad A_x = 0, \quad A_y = B(t)x, \]

\[ B'(t) = \frac{\partial A_y}{\partial x} = B(t), \quad E'(t) = -\frac{\partial A_y}{\partial t} = -\frac{x}{c} \frac{\partial B}{\partial t}, \]

\[ \dot{\kappa}_2 = \dot{\kappa}(x) = -i \frac{\partial}{\partial y} + \frac{e}{\hbar c} B(t)x, \quad \dot{H}_\kappa \to \dot{H}_\kappa(x,t)\kappa(x). \]

\( \lambda(x,t) \approx \text{const. within a unit cell: adiabatic and quasiclassic.} \)
Now the magnetic state is not any more in a closed position space without boundary (unit cell torus or Born-van Karman torus $T^2$); it is bounded in x-direction by edges.

Since outside the edges the vacuum state has $C_1 = 0$ and the Chern number is topological proctected as long as the electronic spectrum is gapped, $C_1 \neq 0$ in the QHE strip can only happen, if there are gapless edge states.


gave a general prescription how automatically an analogous dimension reduction form a fundamental $(2r + 1)D$
Chern-Simons field theory to an adiabatic and quasi-classical
$((2r + 1) − s)D$ Chern-Simons field theory is obtained.

Start with the $(2r + 1)D$ Chern-Simons action

$$S_{\text{(2r+1)}}^{(2r+1)} = \frac{\hbar}{2\pi} \frac{1}{s! (r+1)! (2\pi)^D} \int d^{2r+1} q \, \delta^{A_0} A_r, A_s,$$

$$\cdot \, A_0 \partial A_r \cdots \partial A_{r-4} A_{2s} \text{tr}(D_{A_{2s+1}} A_{2s+2} \cdots D_{A_{2s+1}} A_{2s+2})$$

remove a term $\partial A_r \cdots A_{2s}$, remove the integrations over $dx^{2r} dk_{2r}/(2\pi)$, multiply by the original power $r$ of the external potential $A$, after $s$ reduction steps find

$$S_{\text{(2r+1)}}^{(2r+1-s)} = \frac{\hbar}{2\pi} \frac{1}{s! (r+1-s)! (2\pi)^D} \int d^{2r+1-s} q \, \delta^{A_0} A_r, A_s,$$

$$\cdot \, A_0 \partial A_r \cdots \partial A_{2s-4} A_{2s-2} \text{tr}(D_{A_{2s-1}} A_{2s-2} \cdots D_{A_{2s-1}} A_{2s-2})$$

There are at most $r$ descendants from a fundamental $(2r + 1)D$
Chern-Simons theory, then there is no external gauge potential
$A$ left any more for reduction.

To arrive in our $D \leq 3$-dimensional physical position spaces, the following cases are possible:

$$S_{\text{int}}^{(0+1)} \rightarrow S_{\text{int}}^{(1+1)} \rightarrow S_{\text{int}}^{(2+1)}$$

$$S_{\text{int}}^{(3+1)} \rightarrow S_{\text{int}}^{(4+1)} \rightarrow S_{\text{int}}^{(5+1)}$$

dHvA, QHE, TRI, QSHE
A Chern-Simons theory, depending on the properties of the kinetic terms, may be $C$-invariant for $r$ odd and $T$-invariant for $r$ even.

This is why non-trivial $S^{(2r+1)}_{\text{int}}$ and $S^{(2r+1)}_{\text{ext}}$ must be time-reversal invariant. (In the first (second) descendant of a fundamental $(2r + 1)$-dimensional case an adiabatic quasiclassical polarization (gauge curvature) appears in the action instead of one (two) external potential factor(s) with the same transformation properties as the latter.)

In order to have a topological protection of a non-trivial Chern number state in a dimension reduced Chern-Simons theory, an additional symmetry is needed which enforces the boundary states to be gapless.

Such a symmetry may be one of the ubiquitous discrete symmetries of the kinetic terms in the action:

- **Charge conjugation:** $C^\dagger H C = -H$, $C^\dagger C = C^* C = 1$,
- **Time reversal:** $T^\dagger H T = H$, $T^\dagger T = -T^* T = 1$.

$$
C : A_\mu \to -A_\mu, \quad T : A_\mu \to \left\{ \begin{array}{ll} A_0 & \mu = 0 \\ -A_i & \mu = 1 \end{array} \right.
$$

Chern-Simons Lagrangian $L^{(2r+1)}_{\text{int}}$:

$$
C : L^{(2r+1)}_{\text{int}} \to (-1)^{r+1} L^{(2r+1)}_{\text{int}}, \quad T : L^{(2r+1)}_{\text{ext}} \to (-1)^r L^{(2r+1)}_{\text{ext}}.
$$
Chern-Simons action integrals are integrals of external forms and hence independent of a metric: topological terms.

The QHE is the prototype of such a theory in solid state physics.

A Chern number is the $2r$-dimensional integral of a $2r$-form (Chern form) which is an invariant polynomial of the Berry-Simon curvature form. It locally derives from a Chern-Simons form obtained from the Berry-Simon connection form, but does not depend on the choice of the connection (Berry gauge invariance).

The theory combines a Berry phase gauge theory with a gauge field theory of an external field.

Thank you for your attention!